J Math Chem (2011) 49:12–34
DOI 10.1007/s10910-010-9729-8

ORIGINAL PAPER

Minimum uncertainty wavelets in non-relativistic super-symmetric quantum mechanics

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Received: 2 February 2010 / Accepted: 2 September 2010 / Published online: 23 September 2010

Abstract We consider the connection to the harmonic oscillator, super-symmetric quantum mechanics (SUSY-QM) and coherent states of the recently derived constrained Heisenberg "minimum uncertainty" (μ-) wavelets [Phys Rev Lett 85:5263 (2000); Phys Rev A65: 052106-1 (2002); J Phys Chem A107:7318 (2003)]. We analyze several new types of raising and lowering operators,which also can be viewed as arising from a (non-unitary) similarity transformation of the Harmonic Oscillator Hamiltonian and ladder operators. We show that these new ladder operators lead to a new SUSY formalism for harmonic oscillation, so that the μ-wavelets naturally manifest SUSY properties. Using these new ladder operators, we construct coherent and supercoherent states for the oscillator. In the discussion, we consider possible implications of similarity transformations for quantum mechanics. In an appendix we consider the classical limit of the μ-wavelet oscillator.

Keywords μ-wavelets · Similarity transformations · Super-symmetric quantum mechanics · Coherent states · Ladder operators · Fermion sector · Boson sector

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1 Introduction

This paper is the fourth in a series of studies of constrained minimum Heisenberg uncertainty states (the minimum uncertainty (μ-) wavelets and the related Hermite Distributed Approximating Functionals (HDAFs). Of particular interest here are the fundamental connections of μ-wavelets to the harmonic oscillator and Supersymmetric Quantum Mechanics (SUSY-QM). We discuss these in greater detail below. However, it is useful to begin with some general remarks regarding SUSY-QM, since its relevance to chemical physics is much less clear. The original setting for SUSY was high energy physics and field theory [1–3]. The theory provides a possible pathway towards unifying all of the "elementary particles" and interactions, including gravity. A hallmark of SUSY is the appearance of "sparticles", which basically is a consequence of the fact that, in SUSY, all fundamental particles occur in pairs, one being a boson and the other a fermion (i.e., they differ in spin by $\hbar/2$). In the basic SUSY theory, the sparticles are isoenergetic, so that the energies of the two "sector" Hamiltonians are the same, with the exception of the ground state of the boson sector. In the so-called "good SUSY" the boson ground state has zero energy and there is no corresponding fermion state. All of this seemingly would have nothing to do with problems in chemical physics. However, it has been known for some time that the SUSY ideas can also be applied to non-relativistic quantum systems [4,5]. Until now this has been primarily of academic interest, although it does enable one to obtain analytical ground state wavefunctions for the boson sector (for one dimensional systems). This comes about through the intimate connection of SUSY-QM and the ladder operator approach to the harmonic oscillator (see more below).

At any rate, the remarkable thing is that in SUSY-QM, in addition to the "bosonic" and "fermionic" Hamiltonian partners, in certain cases one can introduce a hierarchy of Hamiltonians [4,5]. All of these share a common spectrum, except that the ground state of each succeeding member of the hierarchy is degenerate with a succeeding excited state of the original Hamiltonian. That is, the ground state of the sector 2 Hamiltonian has the same energy as the first excited state of the (original) sector 1 Hamiltonian. The ground state of the sector 3 Hamiltonian is degenerate with the first excited state of the second sector Hamiltonian and with the second excited state of the sector 1 Hamiltonian. The hierarchy of sector Hamiltonians continues up to the last excited (bound state) of sector 1. Remarkably, this means that excitation energies can be calculated for the sector 1 Hamiltonian by solving a succession of *nodeless* ground states of the higher sector Hamiltonians. It was observed by Kouri, Bittner and co-workers [6–9] that this could provide much more accurate excitation energies using the standard Rayleigh-Ritz variational method (RRVM), but applied successively to obtain ground states. That is, one avoids having to calculate relative phases of the wavefunctions to high accuracy in this approach [8]. Similarly, Bittner, Kouri and co-workers showed that this could also be applied in the context of the Quantum Monte Carlo (QMC) method, since one is always calculating ground state energies of nodeless wavefunctions [7].

Finally, the SUSY-QM approach also leads to a simple, *linear* method to generate the excited state wavefunctions from the hierarchy of ground state wavefunctions.

This is achieved by the use of "charge operators" (analogous to the harmonic operator raising and lowering operators) that transfer wavefunctions between sectors [4–9].

In light of the newly discovered potential significance of SUSY-QM for chemical physics, we are continuing our study of the inter-connections among μ-wavelets, HDAFs, the harmonic oscillator and SUSY-QM.

Of course, the harmonic oscillator is fundamental to a vast range of physics, including the electromagnetic field, spectroscopy, solid state physics, coherent state theory, field theory and string theory, as well as SUSY-QM. Key elements in the wide spread relevance of the harmonic oscillator are the raising and lowering ladder operators which are used to factor the system Hamiltonian. SUSY-QM, which serves as a valuable model for introducing concepts such as boson and fermion dynamics, spontaneous symmetry breaking, field theory and string theory [1–3], can be viewed as generalizing the operators used for factorization of the harmonic oscillator Hamiltonian. Although historically, the factorization procedure has its roots in the operator treatment of the harmonic oscillator, it also has played a fundamental role, e.g., in the quantum mechanics of the hydrogen atom and in the theory of angular momentum [10–13]. In all instances, a factorization strategy involving raising (creation) and lowering (annihilation) operators is central.

The strategy of factoring the Schrödinger operator, and thereby the Schrödinger equation is, of course, equivalent to replacing a second order differential equation with first order differential equations, which are typically much easier to solve. Another example of factorization is Dirac's treatment of the relativistic electron, which leads inexorably to the introduction of spin. From this point of view, it is not surprising that such factorizations in SUSY-QM make it possible to introduce spin-like degrees of freedom directly into non-relativistic quantum mechanics [4,5]. In the case of the harmonic oscillator and angular momentum, this basic factorization leads to raising and lowering operators that are Hermitian conjugates of one another. This fact leads to nice features of the associated algebra for the relevant operators. However, the boson and fermion sector states are simply the even and odd parity HO states, respectively. This basic factorization is intimately connected with the fact that the harmonic oscillator Hamiltonian is the sum of squares of operators $\left(\hat{H}_{HO} = \frac{1}{2}\left(\hat{P}^2/m + m\omega\hat{Q}^2\right)\right)$.

The fundamental connection to the Heisenberg uncertainty principle is also immediately apparent because minimum uncertainty states are easily shown to be eigenvectors of the same lowering operator that results from factoring the harmonic oscillator Hamiltonian. In addition, it is clear that the structure of the harmonic oscillator Hamiltonian is the same in either the coordinate or momentum representations, which is also a fundamental consequence of minimum uncertainty, leading to a state which behaves identically (to within a constant) under the action of the position and momentum operators. Consequently, $\hat{H}_{HO}$ commutes with the Fourier transform, which immediately implies that the harmonic oscillator eigenstates (which are non-degenerate in 1 D) are also eigenstates of the Fourier transform operator.

The fact that canonical coherent states are defined as the eigenstates of the lowering operator implies that they also are fundamentally connected to the harmonic oscillator and the electromagnetic field (quantum optics) [14–16]. Since they also minimize the uncertainty product associated with canonically conjugate pairs of observables, we also expect that coherent states should be important in any problem where simultaneous localization in a pair of canonically conjugate observables is important (e.g., in semiclassical dynamics).

All of these properties are unchanged under any unitary transformation [17,18]. Classically, this translates to the form of the equations of motion being unchanged under a canonical transformation [19–22], and generally, these are the sorts of transformations that are normally investigated in quantum mechanics (see, however, reference 17). However, one might well also ask what happens when one carries out a non-unitary similarity transformation in quantum mechanics, and how does this get reflected in the raising and lowering operators, coherent states and SUSY-QM. Classically, this is the question of what happens when one carries out an "extended canonical transformation"[19], which is characterized by a scaling factor (Jacobian) that differs from one. Quantum mechanically, one expects profound effects of such transformations, not only on the mathematical structure of the theory, but on such major properties as the uncertainty product of position and momentum.

First, it is obvious that the transformed raising and lowering operators will no longer be related by Hermitian conjugation [17,18]. The spectrum and commutation relations of the transformed position and momentum operators, and of the raising and lowering operators, will be unchanged, however. In addition, if the similarity transformation does not possess a bounded inverse (as $x \to \pm\infty$), the transformed harmonic oscillator eigenstates will no longer be a Riesz basis [23–25] for Hilbert space. Furthermore, as has been pointed out previously, in general, non-unitary similarity transformations result in non-Hermitian Hamiltonians [26–29]. Like in reference 17, we also consider a system for which PT symmetry is satisfied by the non-Hermitian μ-wavelet harmonic oscillator Hamiltonian.

In references [32,33], the μ-wavelets (and their sums over integer μ-wavelets only, which sums are called Hermite Distributed Approximating Functionals or HDAFs), were shown to be constrained minimum Heisenberg uncertainty solutions, and to be members of a hierarchy of such states, beginning with the Gaussian. Indeed, the μ-wavelets are precisely generalized Gaussians [32,33]. Both the HDAFs and μ-wavelets are solutions of the same first order differential equation, which arises from the constrained minimization of the Heisenberg uncertainty product, differing only in their boundary conditions. The essence of the constrained minimization is to require that the uncertainty in one observable (either position or momentum) be reduced (squeezed) but with the minimum increase in the uncertainty (due to the change in the wavefunction) of the canonically conjugate observable. The constraint essentially guarantees that the minimizing solution cannot also be a Gaussian.

Recently, we reported the initial study of the relation between these μ-wavelets and the harmonic oscillator eigenstates [34,35]. In that work, it was shown that the μ-wavelets can be viewed as being generated from the harmonic oscillator eigenstates by a (non-unitary) similarity transformation applied to the Hamiltonian. Certain mathematical properties of the μ-wavelets were seen to be a consequence of the fact that the inverse transformation was not bounded as $x \to \pm\infty$. Thus, unlike the usual unitary transformations in quantum mechanics, which correspond to canonical transformations (having a Jacobian equal to 1) in classical mechanics [19], this transformation did not preserve either the orthogonality or the completeness of the harmonic oscillator states in Hilbert space. In fact, the "biorthogonal (analysis) vectors" associated with the μ-wavelets are simply Hermite polynomials [34], which do not belong either to Hilbert or Banach space. Rather, they are distributions.

However, this initial study did not explore the detailed implications of the similarity transformation with regard to SUSY-QM, super-coherent states or the existence of two distinct lowering operators.

In the initial study that derived the μ-wavelets from the harmonic oscillator eigenstates, it became obvious that the usual raising and lowering operators were significantly changed due to the non-unitary similarity transformation [34]. First, the transformed raising and lowering operators are no longer related through Hermitian conjugation [17,18,26–31], although they still result in a factored Hamiltonian for the oscillator. It also was found that the new ladder operators could, themselves, be factored so that additional, fractional raising and lowering operators naturally arose [34]. The original transformed raising and lowering operators transformed states within either the even parity eigenstates or within the odd parity eigenstates, while the fractional raising and lowering operators transformed states between the two parities. In the case of the parity-changing operators, there also arose two distinct lowering operators, with one being simply the inverse of the parity-changing raising operator. It appeared that the second lowering operator was superfluous. Thus, the significance of the additional lowering operator was not explicated beyond observing that it was connected with generating the μ-wavelets that are associated with the odd parity harmonic oscillator eigenstates. Furthermore, the "quantum number" associated with these odd symmetry states was found to be a half-odd-integer, while that for the even symmetry states was an integer [34].

These considerations suggest that further study of the μ-wavelets and the related ladder operators is worthwhile. Herein we report the further analysis of these raising and lowering operators and explore their significance further in relation to the harmonic oscillator, supersymmetric (SUSY) quantum mechanics, coherent and super-coherent states and the classical limit of the μ-wavelet oscillator [32,33,36–39].

This paper is organized as follows. In Sect. 2 we review how the new raising and lowering operators result from the Heisenberg uncertainty principle, since this is useful to establish our notation. There are several distinct sets of these operators all of which operators differ from those for the standard harmonic oscillator. In Sect. 3, we discuss the use of the μ-wavelet operators for the harmonic oscillator. Section 4 shows how the full μ-wavelet SUSY structure is naturally obtained, including the supercharge and Witten parity operators. In Sect. 5, we discuss coherent and supercoherent states [14–16,40–42] in the μ-wavelet theory. Finally, in Sect. 6 we discuss the results and indicate future directions of research. In an appendix, we present the classical limit of the μ-wavelet harmonic oscillator.

2 Brief review of the μ-wavelet hierarchy, raising and lowering operators

We begin with a brief summary of the origin of μ-wavelets from Heisenberg's uncertainty principle. The standard minimization of the uncertainty product makes use of the Schwartz inequality, yielding

$$(\Delta x)^2 (\Delta k)^2 = \frac{\langle \phi_0^\sigma | \hat{x}^2 | \phi_0^\sigma \rangle \langle \phi_0^\sigma | \hat{k}^2 | \phi_0^\sigma \rangle}{\langle \phi_0^\sigma | \phi_0^\sigma \rangle} \geq \frac{\left| \langle \phi_0^\sigma | \hat{x}\hat{k} | \phi_0^\sigma \rangle \right|^2}{\langle \phi_0^\sigma | \phi_0^\sigma \rangle^2} = \frac{1}{4} \tag{1}$$

where $\left[\hat{x}, \hat{k}\right] = i\hat{1}$, implying that the set of operators $\left\{\hat{x}, \hat{k}, \hat{1}\right\}$ constitute a Heisenberg-Weyl Lie Algebra [16]. The minimization of this uncertainty product produces the following equation [16,32,33] for the Gaussian with an arbitrary complex constant σ,

$$\hat{x} | \phi_0^\sigma \rangle = -i\sigma^2 \hat{k} | \phi_0^\sigma \rangle \tag{2}$$

This equation defines the Gaussian as an eigenstate of the lowering operator [14–16], $\hat{a}_{HO} \approx \hat{x} + i\sigma^2\hat{k}$. It is easy to see that here the eigenvalue is zero; other eigenvalues correspond to minimum uncertainty coherent states having nonzero average position and momentum. The corresponding raising operator, $\hat{a}_{HO}^{+} \approx \hat{x} - i\sigma^2\hat{k}$, can be obtained in many ways (e.g., from the harmonic oscillator theory or the requirement that the commutator of the raising and lowering operators be proportional to the identity).

Now suppose we begin with this state of absolute minimum uncertainty, $|\phi_0^\sigma\rangle$ and introduce a variation in the state to reduce further the uncertainty in either $\Delta\hat{x}$ or $\Delta\hat{k}$ (i.e., we "squeeze" the state). A new "minimum uncertainty" state (the μ-wavelet and the HDAF) is defined by

$$|\psi_1^\sigma\rangle = |\phi_1^\sigma\rangle + |\phi_0^\sigma\rangle \equiv |\phi_1^\sigma\rangle + |\psi_0^\sigma\rangle \tag{3}$$

In the usual approach to squeezing coherent states, the resulting state, $|\psi_1^\sigma\rangle$, is also a Gaussian and the uncertainties are of the form $\Delta x = \frac{\sigma}{\sqrt{2}}$ and $\Delta k = \frac{1}{\sqrt{2}\sigma}$, so that their product is one half. We, however, insist that the new state $|\psi_1^\sigma\rangle$ cannot be a Gaussian, and note, therefore, that the overall uncertainty product $\Delta x \Delta k$ must increase [32,33]. We constrain the allowed choices for $|\phi_1^\sigma\rangle$ as follows. For the purpose of illustration, we choose to squeeze Δx (the analysis when squeezing Δk is parallel), and we require that the following quantity, Δ_f^2, be constant for all allowed $|\phi_1^\sigma\rangle$:

$$\Delta_f^2 \equiv \frac{\langle \psi_1^\sigma | \hat{x}^2 | \psi_1^\sigma \rangle}{\langle \psi_1^\sigma | \psi_1^\sigma \rangle^2} [\langle \phi_0^\sigma | \hat{k}^2 | \phi_0^\sigma \rangle + \langle \phi_1^\sigma | \hat{k}^2 | \phi_0^\sigma \rangle + \langle \phi_0^\sigma | \hat{k}^2 | \phi_1^\sigma \rangle] \tag{4}$$

The positive definite quantity, Δ_v^2 is then defined by

$$\Delta_v^2 \equiv \frac{\langle \psi_1^\sigma | \hat{x}^2 | \psi_1^\sigma \rangle \langle \phi_1^\sigma | \hat{k}^2 | \phi_1^\sigma \rangle}{\langle \psi_1^\sigma | \psi_1^\sigma \rangle^2} \tag{5}$$

and one easily verifies that [32,33]

$$(\Delta x \Delta k)^2 = \Delta_f^2 + \Delta_v^2 \tag{6}$$

Clearly, since Δ_f^2 is fixed for all possible $|\phi_1^\sigma\rangle$, $(\Delta x \Delta k)^2$ will be a minimum provided Δ_v^2 is a minimum. Since this is greater than 0, the fixed value of Δ_f^2 sets the floor below which $(\Delta x \Delta k)^2$ cannot go. The minimization of Eq. 6 is carried out in a manner identical to what is done for Eq. 1. Note that one is minimizing the full uncertainty in $\hat{x}$, but only the uncertainty in $\hat{k}$ that comes solely from $|\phi_1^\sigma\rangle$. We obtain

$$\left[\hat{x} + i\sigma^2\hat{k}\right]|\phi_1^\sigma\rangle = i\sigma^2\hat{k}|\phi_0^\sigma\rangle \tag{7}$$

Obviously, the operator $\hat{x} + i\sigma^2\hat{k}$, which is, to within a constant factor, the standard lowering operator, is applied to $|\phi_1^\sigma\rangle$, which unlike the state, $|\phi_0^\sigma\rangle$, is not annihilated. From the form of Eq. 7, $i\sigma^2\hat{k}$ appears to be a type of raising operator [34]. Repeating the minimization process, one easily finds for arbitrary indexed ϕ_{j+1}^σ,

$$\left[\hat{x} + i\sigma^2\hat{k}\right]|\phi_{j+1}^\sigma\rangle = i\sigma^2\hat{k}|\phi_j^\sigma\rangle \equiv |\phi_{j<j'<j+1}^\sigma\rangle \tag{8}$$

Since j and $j+1$ are both integers and the operator on the LHS of Eq. 8 is clearly a lowering operator while the operator on the right is apparently a raising operator, this suggests the last equality, where j' is fractional [34]. Following Hoffman and Kouri [32,33], for positive index j, we call the $|\phi_j\rangle$ vectors "μ-wavelets". From Eq. 8, we can now define a new, " μ-wavelet lowering operator" as:

$$\hat{a}_\mu = \left(\frac{-i}{\sigma^2\hat{k}}\right)\hat{x} + 1 \tag{9}$$

(Note that $\hat{a}_\mu$ is the product of $(\frac{-i}{\sigma^2\hat{k}})$ and the "standard" lowering operator, $(\hat{x} + i\sigma^2\hat{k})$) Obviously, we can express this in the $x-$representation as

$$\hat{a}_\mu = \left(\frac{1}{\sigma^2}\partial^{-1}\right)x + 1 \tag{10}$$

where $\partial \equiv \frac{\partial}{\partial x}$, $\partial^{-1} \equiv \left(\frac{\partial}{\partial x}\right)^{-1}$. Strictly for the sake of simplicity, we shall choose the constant of integration for $\left(\frac{\partial}{\partial x}\right)^{-1}$ to be zero. In the $k-$representation, the new lowering operator has the form

$$\hat{a}_\mu = \left(\frac{1}{\sigma^2 k}\right)\frac{\partial}{\partial k} + 1 \tag{11}$$

One easily finds that if $\hat{a}_\mu^+$ is given by

$$\hat{a}_\mu^+ = \frac{\sigma^2\hat{k}^2}{2} \tag{12}$$

then, from Eqs. 2 and 9, we can easily establish the absolute minimum uncertainty state expression,

$$\hat{a}_\mu |\phi_0^\sigma\rangle = 0 \tag{13}$$

Therefore, in the momentum representation, this state is

$$\langle k|0, \sigma, \mu\rangle = \left(\frac{\sigma^2}{\pi}\right)^{\frac{1}{4}} e^{\frac{-\sigma^2 k^2}{2}} \tag{14}$$

where $\langle k|0, \sigma, \mu\rangle \equiv \langle k|\phi_0^\sigma\rangle$.

It is important to stress that this differs from the usual harmonic oscillator ground state, whose Gaussian exponent is $\frac{-\sigma^2 k^2}{4}$; i.e., the usual HO ground state is the square root of the Hermite polynomial generator [34]. The distinction is more apparent when one examines the excited states of the μ-wavelet harmonic oscillator. Similarly, using Eq. 10, we obtain the x-representation ground state,

$$\langle x|0, \sigma, \mu\rangle = \left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{4}} e^{\frac{-x^2}{2\sigma^2}}. \tag{15}$$

It is then easy to construct the n-th momentum representation μ-wavelet state by applying the raising operator, Eq. 12, and requiring the L^2-norm of the state to be unity, yielding

$$\langle k|n, \sigma, \mu\rangle = \frac{\sigma^{2n+\frac{1}{2}}}{\left\{\left(2n - \frac{1}{2}\right)!\right\}^{\frac{1}{2}}} k^{2n} e^{\frac{-\sigma^2 k^2}{2}}. \tag{16}$$

Also, we can express this n-th state in terms of powers of the raising operator $\left(\hat{a}_\mu^+\right)^n$:

$$\langle k|n, \sigma\mu\rangle = \alpha_n (\hat{a}_\mu^+)^n \langle k|0, \sigma, \mu\rangle, \tag{17}$$

where $\alpha_n = \frac{\pi^{1/4} 2^n}{\{(2n-1/2)!\}^{1/2}}$ ensures a unit L^2-norm. One can adjust the energy spacing between the states by defining the raising operator $\hat{\alpha}_\mu^+ = 2\hat{a}_\mu^+$ then

$$\hat{\alpha}_\mu^+ = -\sigma^2 \partial^2 \quad \text{or} \quad \hat{\alpha}_\mu^+ = -\sigma^2 k^2. \tag{18}$$

Note that $\hat{\alpha}_\mu = \hat{a}_\mu$. We remark that the above expressions imply the existence of a fractional raising operator (see the next section). Then Eq. 17 becomes

$$\langle k|n, \sigma, \mu\rangle = C_n (\hat{\alpha}_\mu^+)^n \langle k|0, \sigma, \mu\rangle \tag{19}$$

Because $C_n = \alpha_n/2^n$, we easily find that

$$C_n = \frac{\pi^{1/4}}{\{(2n - \frac{1}{2})!\}^{\frac{1}{2}}} \tag{20}$$

There are, of course, several methods to obtain the coordinate representation of the n-th μ-wavelet [35]. Here, we use the raising operator to generate the n-th state from the ground state, Eq. 19,

$$\begin{aligned}\langle x|n, \sigma, \mu\rangle &= C_n(\hat{\alpha}_{\mu}^{+})^n \langle x|0, \sigma, \mu\rangle \\ &= (-1)^n C_n \sigma^{2n} \partial^{2n} \langle x|0, \sigma, \mu\rangle \\ &= \frac{(-1)^n \sigma^{2n-1/2}}{\{(2n - \frac{1}{2})!\}^{\frac{1}{2}}} e^{\frac{-x^2}{2\sigma^2}} (\partial - \frac{x}{\sigma^2})^{2n} \\ &= \frac{(-1)^n}{2^n \sigma^{1/2} \{(2n - 1/2)!\}^{1/2}} e^{\frac{-x^2}{2\sigma^2}} H_n\left(\frac{x}{\sqrt{2}\sigma}\right)\end{aligned} \tag{21}$$

We again see that the argument of the Hermite polynomial is exactly the square root of the argument of the Gaussian. In the standard harmonic oscillator states, the argument of the Hermite is the square root of twice the argument of the Gaussian. Explicitly, the μ-wavelet and harmonic oscillator states are related by [34]

$$\langle x|n, \sigma, \mu\rangle = (-1)^n \left\{(2\pi)^{1/2} \frac{(2n)!}{(2n - 1/2)!}\right\}^{1/2} e^{\frac{-x^2}{4\sigma^2}} \langle x|2n, \sqrt{2}\sigma, HO\rangle \tag{22}$$

This relationship corresponds to a similarity transformation, connecting the usual harmonic oscillator states to the μ-wavelet states [34]. Clearly, the inverse transform in the x-representation is simply multiplication by $e^{\frac{x^2}{4\sigma^2}}$, which is obviously unbounded on the x-domain $-\infty < x < \infty$. Therefore, this similarity transformation does not produce a Riesz basis from the harmonic oscillator states [23–25,34].

It must also be noted that since the quantum index, n appears on the right hand side of Eq. 22 as $2n$, it is perfectly allowable for n to be a half-odd-integer [34,35]. In the case of half-odd-integer values of n, the state defined by Eq. 22 is

$$\left\langle x|\frac{n}{2}, \sigma, \mu\right\rangle = \frac{(-i)^n}{2^{n/2} \sigma^{1/2} \{(n - 1/2)!\}^{1/2}} e^{\frac{-x^2}{2\sigma^2}} H_n(x/\sqrt{2}\sigma) \tag{23}$$

These half-odd-integer μ-wavelet states are clearly related to the odd parity harmonic oscillator eigenstates, but they will be interpreted, using the structure of supersymmetry, as the elements of the fermion sector. They can be viewed as the consequence of the existence of new half-odd-integer raising and lowering operators for the μ-wavelets.

3 Fractional-wavelet ladder operators and the harmonic oscillator

Having obtained the fundamental definitions of the μ-wavelet ladder operators that preserve the parity of the state to which they are applied, we now show how the parity changing (fractional) ladder operators naturally arise.. In the standard theory, one factors the harmonic oscillator Hamiltonian by introducing the raising and lowering "factor" operators. For μ-wavelet oscillators, we note that, from Eq. 12 or 18, we can factor the raising operator, $\hat{\alpha}_\mu^+$, by defining $\hat{a}_{1/2}^+$ such that [35]

$$\hat{\alpha}_\mu^+ = \sqrt{\hat{\alpha}_\mu^+ \hat{\alpha}_\mu^+} = \hat{a}_{1/2}^+ \hat{a}_{1/2}^+ \tag{24}$$

so that we have

$$\hat{a}_{1/2}^+ = \sqrt{\hat{\alpha}_\mu^+} = \sigma \hat{k} \tag{25}$$

Requiring that the commutator relation $[\hat{a}_{1/2}, \hat{a}_{1/2}^+] = 1$ be satisfied, the corresponding lowering operator is found to be

$$\hat{a}_{1/2} = -i \left(\frac{\hat{x}}{\sigma} + i\sigma\hat{k} \right), \tag{26}$$

which also results from multiplying Eq. 8 by $-\frac{i}{\sigma}$. This establishes the conjecture that these equations imply the existence of another state between that labeled by j and that labeled by $j+1$. We note that $\hat{a}_{1/2}$ differs by a phase from the usual harmonic oscillator lowering operator, and it is now paired with the raising operator $\hat{a}_{1/2}^+$. With the help of Eq. 11, it is also convenient to factor the lowering operator, $\hat{\alpha}_\mu$ by defining

$$\hat{\alpha}_\mu \equiv \hat{\hat{a}}_{1/2} \hat{a}_{1/2} \tag{27}$$

where the operator

$$\hat{\hat{a}}_{1/2} = \frac{1}{\sigma \hat{k}} \tag{28}$$

was introduced earlier in references [34,35]. For simplicity, we again take the constant of integration to be zero. There are now two distinct lowering operators that occur as factors to produce $\hat{\alpha}_\mu$, reflecting the facts that one allows the satisfaction of a commutator condition with the raising operator, $\hat{a}_{1/2}^+$, and the other is the inverse of $\hat{a}_{1/2}^+$. Such an inverse corresponds to a different sort of lowering operation, of course, and it implies that an anti-commutator relation appropriate for fermions will be satisfied. As might be expected, these new operators play an important role in the supersymmetric coherent state structure in the μ-wavelet theory.

Similar to the situation of the standard harmonic oscillator, these operators raise or lower the μ-wavelet quantum index, but now by $\frac{1}{2}$ rather than by 1. Then the ground state is denoted by $|0\rangle$, the first excited state by $|1/2\rangle$, etc.

For the raising operators, one can show that

$$\hat{a}^+_{1/2}|n\rangle = (2n + 1/2)^{\frac{1}{2}}|n + 1/2\rangle \tag{29}$$

$$\hat{\alpha}^+_\mu|n\rangle = \sqrt{(2n + 1/2)(2n + 3/2)}|n + 1\rangle \tag{30}$$

The above equations are consistent with the original definitions, Eq. 11 and 29. For the lowering operators, we find

$$\hat{a}_{1/2}|n\rangle = \frac{2n}{(2n - 1/2)^{1/2}}|n - 1/2\rangle \tag{31}$$

$$\hat{\hat{a}}_{1/2}|n\rangle = \frac{1}{(2n - 1/2)^{1/2}}|n - 1/2\rangle \tag{32}$$

$$\hat{\alpha}_\mu|n\rangle = \frac{2n}{\sqrt{(2n - 1/2)(2n - 3/2)}}|n - 1\rangle \tag{33}$$

One may define two distinct μ-wavelet "harmonic oscillator" Hamiltonians;

$$\hat{H}_B|n\rangle_B = \hat{\alpha}^+_\mu\hat{\alpha}_\mu|n\rangle_B = 2n|n\rangle_B = \hat{a}^+_{1/2}\hat{a}^+_{1/2}|n\rangle_B \tag{34}$$

where here, the index n of $|n\rangle_B$ takes on only even values: $n = 0, 2, 4, \ldots, 2n$, and

$$\hat{H}_F|n\rangle_F = \hat{\alpha}_\mu\hat{\alpha}^+_\mu|n\rangle_F = 2(n + 1)|n\rangle_F \tag{35}$$

where here the index n of $|n\rangle_F$ takes on only odd values: $n = 1, 3, 5, \ldots, 2n + 1$. In this description, the μ-wavelet oscillator has a double degeneracy for each level except for the zero energy ground state. We see then that

$$\hat{H}_F = \hat{H}_B + 2 \tag{36}$$

This immediately shows that the μ-wavelets naturally admit a supersymmetric structure [2–5,36–39], like that of the standard harmonic oscillator. Here $\hat{H}_B$ is identified as the boson sector Hamiltonian and $\hat{H}_F$ is identified as the fermion sector Hamiltonian. Because of Eq. 36, this corresponds to good SUSY. However, rather than use the subscripted state notation, $|n\rangle_B$ and $|n\rangle_F$, we can simply replace n with $n/2$. Then, we have the half-odd-integer states, satisfying

$$\hat{a}^+_{1/2}|n/2\rangle = (n + 1/2)^{1/2}\left|\frac{n+1}{2}\right\rangle \tag{37}$$

$$\hat{a}_{1/2}|n/2\rangle = n\hat{\hat{a}}_{1/2}|n/2\rangle = \frac{n}{(n - 1/2)^{1/2}}\left|\frac{n-1}{2}\right\rangle \tag{38}$$

The two possible Hamiltonians for the half-odd-integer states are given by

$$\hat{H}^{(1)}_{1/2}\left|\frac{n}{2}\right\rangle = \hat{a}^{+}_{1/2}\hat{a}_{1/2}\left|\frac{n}{2}\right\rangle = n\left|\frac{n}{2}\right\rangle \tag{39}$$

$$\hat{H}^{(2)}_{1/2}\left|\frac{n}{2}\right\rangle = \hat{a}_{1/2}\hat{a}^{+}_{1/2}\left|\frac{n}{2}\right\rangle = (n+1)\left|\frac{n}{2}\right\rangle \tag{40}$$

In the case of SUSY-QM, the above Eqs. 39 and 40 show that $\hat{a}_{1/2}$ and $\hat{a}^{+}_{1/2}$ provide an interesting alternative to the $\hat{\alpha}_\mu, \hat{\alpha}^{+}_\mu$ for the μ-wavelet states. In turn, these also differ from the usual harmonic oscillator raising and lowering operators, $\hat{a}^{+}$ and $\hat{a}$; see Eqs. 9 and 12. Note that $\hat{\hat{a}}_{1/2}$ is just the inverse of the raising operator, $\hat{a}^{+}_{1/2}$, and its effect on the state $|\frac{n}{2}\rangle$ is also dependent upon the lowering operator, $\hat{a}_{1/2}$, as can be seen in Eq. 38. On the surface, it would appear that the $\hat{a}_{1/2}$ and $\hat{a}^{+}_{1/2}$ operators are sufficient to describe all of the μ-wavelet states without need for $\hat{\hat{a}}_{1/2}$. The operator $\hat{a}_{1/2}$ annihilates the ground state, while $\hat{\hat{a}}_{1/2}$ does not annihilate the ground state. However, in the momentum space, $\hat{\hat{a}}_{1/2}$ is simpler to use to generate the lower states than $\hat{a}_{1/2}$. In the coordinate space, $\hat{a}_{1/2}$ is easier to use to generate the states than $\hat{\hat{a}}_{1/2}$ because $\hat{\hat{a}}_{1/2}$ is an integral operator in this representation.

We summarize the relationships among these operators associated with the μ-wavelets as follows

$$\left[\hat{a}_\mu, \hat{a}^{+}_\mu\right] = \left[\hat{\alpha}_\mu, \hat{\alpha}^{+}_\mu\right] = \left\{\left[\hat{\hat{a}}_{1/2}\hat{a}_{1/2}, \hat{a}^{+}_{1/2}\right], \hat{a}^{+}_{1/2}\right\} = \left\{\hat{\hat{a}}_{1/2}, \hat{a}^{+}_{1/2}\right\} = 2 \tag{41}$$

$$\left[\hat{a}_{1/2}, \hat{a}^{+}_{1/2}\right] = 1 \tag{42}$$

$$\begin{aligned}\left[\hat{a}_\mu, \hat{a}_\mu\right] &= \left[\hat{a}^{+}_\mu, \hat{a}^{+}_\mu\right] = \left[\hat{a}_{1/2}, \hat{a}_{1/2}\right] = \left[\hat{a}^{+}_{1/2}, \hat{a}^{+}_{1/2}\right] = \left[\hat{\hat{a}}_{1/2}, \hat{a}_{1/2}\right] \\ &= \left[\hat{\hat{a}}_{1/2}, \hat{a}^{+}_{1/2}\right] = 0\end{aligned}$$

As usual, {, } denotes the anticommutator.

4 The full supersymmetric structure of the μ-wavelet harmonic oscillator

As is clear from Eqs. 34–36, the eigenenergies of the μ-wavelets possess the SUSY structure [2,3,36–39]. To explicate this, we briefly outline the SUSY approach to the ordinary Harmonic Oscillator. We recall that the harmonic oscillator satisfies the well known Schrödinger equation:

$$\left\{\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right\}\psi = \varepsilon\psi. \tag{43}$$

As is customary, we simplify this to

$$\hat{H}_{HO}\psi = (-\partial^2 + x^2)\psi = \varepsilon\psi \tag{44}$$

where, $x \to x\sqrt{m\omega/\hbar}$, $\varepsilon \to 2\varepsilon/\hbar\omega$. The original boson (B(O)) SUSY harmonic oscillator Hamiltonian $\hat{H}_{B(O)}$ and the original fermion (F(O)) SUSY harmonic oscillator Hamiltonian $\hat{H}_{F(O)}$ are well-known to be

$$\hat{H}_{B(O)} = \hat{a}^+\hat{a} = (-\partial + x)(\partial + x) = H_{HO} - 1 \tag{45}$$

$$\hat{H}_{F(O)} = \hat{a}\hat{a}^+ = (\partial + x)(-\partial + x) = H_{HO} + 1 \tag{46}$$

$$\hat{H}_{B(O)}|n\rangle = \hat{a}^+\hat{a}|n\rangle \quad \hat{H}_{F(O)}|n\rangle = \hat{a}\hat{a}^+|n\rangle = 2(n+1)|n\rangle \tag{47}$$

$$\hat{H}_{HO}|n\rangle = (\hat{a}^+\hat{a} + 1)|n\rangle = (2n+1)|n\rangle \tag{48}$$

Therefore,

$$\hat{H}_{F(O)} = \hat{H}_{B(O)} + 2; \tag{49}$$

Clearly the μ-wavelet Hamiltonians, $\hat{H}_B$, $\hat{H}_F$ Eqs. 34–36, obey similar equations as those of the original SUSY Hamiltonians, Eqs. 45–49. It is interesting to explore the details of the μ-wavelet SUSY. We treat the states of integer quantum indices as the boson sector and the states with the half-odd-integer quantum indices as the fermion sector. Then the boson sector Hamiltonian is

$$\hat{H}_B = \hat{\alpha}_\mu^+\hat{\alpha}_\mu = \hat{a}_{1/2}^+\hat{a}_{1/2}, \tag{50}$$

and we note that the new fractional lowering operator, $\hat{\tilde{a}}_{1/2}$, is immediately eliminated due to the ordering of the $\hat{\alpha}_\mu^+$ and $\hat{\alpha}_\mu$ in the boson sector Hamiltonian. The corresponding fermion sector Hamiltonian is

$$\hat{H}_F = \hat{\alpha}_\mu\hat{\alpha}_\mu^+ = \hat{a}_{1/2}^+\hat{a}_{1/2} - 1 = \hat{H}_B - 1 \tag{51}$$

where use of the commutator $[\hat{a}_{1/2}, \hat{a}_{1/2}^+] = 1$ is required to eliminate $\hat{\tilde{a}}_{1/2}$ in the fermion sector Hamiltonian. Thus, it appears that the $\hat{\tilde{a}}_{1/2}$ ladder operator is not required in the SUSY theory of the μ-wavelet oscillator. This curious state of affairs will be the subject of a subsequent investigation [43].

Now, let us consider the good SUSY μ-wavelet Hamiltonians further. Using Eqs. 50,51, the good SUSY Schrödinger equations for the two sectors are

$$\hat{H}_B|n\rangle = \hat{a}_{1/2}^+\hat{a}_{1/2}|n\rangle = 2n|n\rangle \tag{52}$$

$$\hat{H}_{F(G)}|n+1/2\rangle = \hat{a}_{1/2}^+\hat{a}_{1/2}|n+1/2\rangle = 2\,(n+1/2)\,|n+1/2\rangle \tag{53}$$

Using Eqs. 52 and 53, the good SUSY Hamiltonian and Schrödinger equation are

$$\hat{H} = \begin{pmatrix} \hat{H}_B & 0 \\ 0 & \hat{H}_{F(G)} \end{pmatrix}, \tag{54}$$

and

$$\hat{H}\begin{pmatrix} |n\rangle \\ |n-1/2\rangle \end{pmatrix} = 2n\begin{pmatrix} |n\rangle \\ |n-1/2\rangle \end{pmatrix}, \quad n \geq 1. \tag{55}$$

When $n = 0$, the state is defined as $\binom{|0\rangle}{0}$, because the state $|-1/2\rangle = 0$. In SUSY-QM, a fundamental feature is the expression of the Hamiltonian in terms of the so-called "super-charge operators". These have the property that their anti-commutator equals the SUSY Hamiltonian. The corresponding μ-wavelet supercharge operators are defined by

$$\hat{Q} \equiv \begin{pmatrix} 0 & 0 \\ \hat{a}_{1/2} & 0 \end{pmatrix}, \quad \hat{Q}^{+} \equiv \begin{pmatrix} 0 & \hat{a}_{1/2}^{+} \\ 0 & 0 \end{pmatrix} \tag{56}$$

The anticommutators of these operators are

$$\{\hat{Q}, \hat{Q}^{+}\} = \begin{pmatrix} \hat{a}_{1/2}^{+}\hat{a}_{1/2} & 0 \\ 0 & \hat{a}_{1/2}\hat{a}_{1/2}^{+} \end{pmatrix} = \hat{H} \tag{57}$$

$$\{\hat{Q}, \hat{Q}\} = \{\hat{Q}^{+}, \hat{Q}^{+}\} = 0 \tag{58}$$

The above equations obviously constitute a SUSY algebra, which satisfies Lie's super-algebra. To verify conservation of supercharge [36–39], we compute the commutators of $\hat{Q}$ and $\hat{Q}^{+}$ with the SUSY Hamiltonian:

$$[\hat{Q}, \hat{H}] = [\hat{Q}, \hat{Q}\hat{Q}^{+} + \hat{Q}^{+}\hat{Q}] = \hat{Q}\left[\hat{Q}, \hat{Q}^{+}\right] + [\hat{Q}, \hat{Q}^{+}]\hat{Q} = 0 \tag{59}$$

Similarly, one has

$$[\hat{Q}^{+}, \hat{H}] = 0 \tag{60}$$

Because charge is conserved, this Hamiltonian contains coordinates which are quantized by commutators and anticommutators. Obviously, Eq. 56 can be expressed in the form

$$\hat{Q} = \hat{a}_{1/2}\hat{q}^{+} = \left(-i\frac{\hat{x}}{\sigma} + \sigma\hat{k}\right)\hat{q}^{+} \quad \hat{Q}^{+} = \hat{a}_{1/2}^{+}q = \sigma\hat{k}\hat{q} \tag{61}$$

where,

$$\hat{q}^{+} = \hat{\sigma}_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{q} = \hat{\sigma}_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{62}$$

$$[\hat{q}^{+}, \hat{q}] = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = -\hat{\sigma}_{z} \tag{63}$$

$$\left\{\hat{q}, \hat{q}^{+}\right\} = 1 \quad \{\hat{q}, \hat{q}\} = \{\hat{q}^{+}, \hat{q}^{+}\} = 0 \tag{64}$$

As usual, [,] denotes the commutator and {, } the anticommutator. and the $\hat{\sigma}$'s are Pauli matrices. Thus, using Eq. 57 or 61, we obtain the Hamiltonian of the supersymmetric μ-wavelets.

Because these energy eigenvalues are $2n$, it is convenient to scale the Hamiltonian by a factor of $1/2$:

$$\hat{H}^{B}_{\mu-\text{wavelet}} = \frac{1}{2}(\sigma^2\hat{k}^2 - i\hat{k}\hat{x}) = \frac{1}{2}\hat{H}_B \tag{65}$$

$$\hat{H}^{F(G)}_{\mu-\text{wavelet}} = \frac{1}{2}(\sigma^2\hat{k}^2 - i\hat{x}\hat{k}) = \frac{1}{2}\hat{H}_F \tag{66}$$

In this form, the transition from the quantum to the classical limit is easily carried out as we discuss it in detail in the Appendix (of course, in the classical limit, the boson and fermion sector Hamiltonians yield the same result, namely $H_{\mu-\text{wavelet}}$). Also, we can express these Hamiltonians, using the half-integer raising and lowering operators, according to

$$\hat{H}^{\pm} = \frac{1}{2}\left\{\hat{a}^{+}_{1/2}, \hat{a}_{1/2}\right\} \pm \frac{1}{2}\left[\hat{a}^{+}_{1/2}, \hat{a}_{1/2}\right], \tag{67}$$

where the $+$ sign denotes fermions and the $-$ sign denotes bosons. We interpret the Hamiltonian forms of Eqs. 65 and 66 as implying that the potential of the μ-wavelet harmonic oscillator depends upon both the velocity and position while the original harmonic oscillator potential depends upon the position only. Even though Eqs. 65 and 66 are the same in the limit of classical mechanics, they differ by one energy unit in quantum mechanics. Note that at this stage, the extra lowering operator, $\hat{\hat{a}}_{1/2}$, appears to be completely unnecessary!

From the viewpoint of SUSY-QM, it is clear from Eq. 66 that the superpotential of the μ-wavelets is the same as the harmonic oscillator's "shape invariant" potential. Like the supercharge operator, the Witten parity [34], $\hat{W}$, of the SUSY μ-wavelets is conserved. The self-adjoint operator, $\hat{W}$ obeys the relations

$$\hat{W} = \frac{2}{\hat{H}}\hat{Q}\hat{Q}^{+} - 1 = \frac{[\hat{Q}, \hat{Q}^{+}]}{\{\hat{Q}, \hat{Q}^{+}\}} \tag{68}$$

$$[\hat{W}, \hat{H}] = 0 \quad \{\hat{W}, \hat{Q}\} = \{\hat{W}, \hat{Q}^{+}\} = 0 \quad \hat{W}^2 = 1 \tag{69}$$

For our μ-wavelets, we find directly that

$$\hat{W} = \frac{[\hat{Q}, \hat{Q}^{+}]}{\{\hat{Q}, \hat{Q}^{+}\}} = \frac{\begin{pmatrix} -\hat{a}^{+}_{1/2}\hat{a}_{1/2} & 0 \\ 0 & \hat{a}_{1/2}\hat{a}^{+}_{1/2} \end{pmatrix}}{\begin{pmatrix} \hat{a}^{+}_{1/2}\hat{a}_{1/2} & 0 \\ 0 & \hat{a}_{1/2}\hat{a}^{+}_{1/2} \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{70}$$

The Witten parity is related to the fermion number operator, $\hat{F}$, defined as

$$\hat{F} = \hat{f}^{+}\hat{f} = \hat{q}^{+}\hat{q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{71}$$

where $\hat{f}^{+} = \hat{a}^{+}_{1/2}\hat{q}^{+}$, and $\hat{f} = \hat{\hat{a}}_{1/2}\hat{q}$. Because of the relation $\hat{W} = 2\hat{F} - 1$, the eigenspace of $\hat{W}$ with eigenvalue, $+1$, ($\hat{F} = 1$) is the fermionic subspace, and that of the -1 eigenvalue ($\hat{F} = 0$) is the bosonic subspace. The SUSY transformation with the supercharge operators connects the fermion and boson states:

$$\hat{Q}\begin{pmatrix} |n, B\rangle \\ |n - 1/2, F\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{a}_{1/2}|n, B\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2n}{(2n-1/2)^{1/2}}|n - 1/2, F\rangle \end{pmatrix}$$
$$\hat{Q}^{+}\begin{pmatrix} |n, B\rangle \\ |n - 1/2, F\rangle \end{pmatrix} = \begin{pmatrix} \hat{a}^{+}_{1/2}|n - 1/2, F\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} (2n - 1/2)^{1/2}\,|n, B\rangle \\ 0 \end{pmatrix}$$

Clearly, $\hat{Q}$ and $\hat{Q}^{+}$, as well as $\hat{a}_{1/2}$ and $\hat{a}^{+}_{1/2}$ play analogous roles in the transformation from the boson sector to the fermion sector.

5 Coherent and supercoherent state theory for μ-wavelets

Coherent states can be viewed in various ways, [14–16], including as minimum uncertainty states, as eigenstates of an annihilation operator, and as displacement operator coherent states. Like the original SUSY harmonic oscillator, the μ-wavelets can be considered to involve both bosonic and fermionic degrees of freedom. The displacement operator approach has been discussed in detail using a group-theoretical treatment by Perelomov [16]. Our starting point is to construct the μ-wavelet coherent state theory using the fractional annihilation operator, $\hat{a}_{1/2}$ rather than $\hat{\alpha}_{\mu}$, because $\hat{\alpha}_{\mu}$ is a composite of the two fractional lowering operators, $\hat{a}_{1/2}, \hat{\hat{a}}_{1/2}$ for the μ-wavelet, (see Eq. 27). To obtain the coherent states [14–16] for the μ-wavelets, we start by defining them according to the usual sort of eigenvalue equation:

$$\hat{a}_{1/2}|\alpha\rangle = \alpha|\alpha\rangle, \tag{72}$$

where α is a complex number given by

$$\alpha = -i\left(\frac{x}{\sigma} + i\sigma k\right). \tag{73}$$

To solve the Eq. 72, we follow Perelomov [16];

$$\begin{aligned} |\alpha\rangle &= e^{\alpha\hat{a}^{+}_{1/2} - \beta\hat{a}_{1/2}|}|0\rangle \\ &= e^{\frac{-\alpha\beta}{2}} e^{\alpha\hat{a}^{+}_{1/2}}|0\rangle \quad \beta = \sigma k \end{aligned}$$

$$= e^{\frac{-\alpha\beta}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}_{1/2}^+)^n}{n!} |0\rangle$$
$$= e^{\frac{-\alpha\beta}{2}} \sum_{n=0}^{\infty} \frac{\pi^{1/4} \alpha^n \{(n-1/2)!\}^{1/2}}{n!} \left|\frac{n}{2}\right\rangle. \tag{74}$$

We know that the terms in the sum for which n is even belong to the boson sector and those for which n is odd belong to the fermion sector; that is,

$$|\alpha\rangle_B = |\alpha\rangle_{n=even\ integers} \quad |\alpha\rangle_F = |\alpha\rangle_{n=odd\ integers}. \tag{75}$$

Thus, the basic μ-wavelet coherent state, $|\alpha\rangle$, is a sum of $|\alpha\rangle_B$ and $|\alpha\rangle_F$. Now we proceed to construct a μ-wavelet supercoherent state in a simple way. One can readily establish the following relations:

$$\hat{a}_{1/2}|\alpha\rangle_B = \alpha|\alpha\rangle_F \quad \hat{a}_{1/2}|\alpha\rangle_F = \alpha|\alpha\rangle_B. \tag{76}$$

It follows that we can define a "super-lowering operator", $\hat{A}$, as $\hat{A} = \begin{pmatrix} 0 & \hat{a}_{1/2} \\ \hat{a}_{1/2} & 0 \end{pmatrix}$. The super-coherent boson sector is $|\alpha\rangle_B$ and its fermion sector is $|\alpha\rangle_F$. It is then easily seen that

$$\hat{A} \begin{pmatrix} |\alpha\rangle_B \\ |\alpha\rangle_F \end{pmatrix} = \alpha \begin{pmatrix} |\alpha\rangle_B \\ |\alpha\rangle_F \end{pmatrix}. \tag{77}$$

It is also natural to define $\hat{A}^+$ as

$$\hat{A}^+ = \begin{pmatrix} 0 & \hat{a}_{1/2}^+ \\ \hat{a}_{1/2}^+ & 0 \end{pmatrix}. \tag{78}$$

It is again easily seen that the commutator $[\hat{A}, \hat{A}^+] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then the μ-wavelet super-symmetric displacement operator is

$$\hat{D}(\alpha, \beta) = e^{\alpha\hat{A}^+ - \beta\hat{A}} = e^{\alpha\beta/2} e^{\alpha\hat{A}^+} e^{-\beta\hat{A}} \tag{79}$$

This is applied to the SUSY μ-wavelet ground state, $\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}$, which produces the desired super-coherent state. Two interesting questions to be studied in subsequent research are: (1) What are the coherent states generated by the composite operators $\hat{\alpha}_\mu$ and $\hat{\alpha}_\mu^+$? (2) What is the role of the additional lowering operator $\widehat{\hat{a}}_{1/2}$?

6 Discussion

6.1 The μ-wavelet oscillator versus the harmonic oscillator

Although the fundamental origin of the μ-wavelet is from a constrained minimization of the Heisenberg uncertainty product for the canonical position and momentum, they can also be viewed as resulting from a (non-unitary) similarity transformation of the harmonic oscillator. As is shown in the Appendix, the classical μ-wavelet oscillator dynamics is also harmonic but with a modified frequency and time dependence of the phase of the momentum, compared to the original harmonic oscillator. Quantum mechanically, the energy and commutation relations are unchanged, but additional ladder operators occur. A striking difference is also in the fact that the uncertainty product for the μ-wavelet oscillator states increases much more slowly with quantum number than the original HO. In particular, the uncertainty product for the nth harmonic oscillator state varies as the first power of n, while that of the μ-wavelet oscillator [32] varies as $n^{1/2}$. Indeed, the uncertainty for sums of the first $M\mu$-wavelets varies as $M^{1/4}$. The uncertainty of the 300th HO state is 300.5 while that of the μ-wavelet with $M = 300$ is of the order of 17. This suggests interesting avenues for further research using the μ-wavelet ladder operators and Hamiltonians. In much research on the dynamics of systems immersed in a bath, the bath is modeled by an assembly of harmonic oscillators (e.g., see the work of Caldeira and Leggett [44]). It may be interesting to study such problems in which the usual HO ladder operators are replaced by either of the pairs [45] $\hat{a}_\mu^+, \hat{a}_\mu$ or $\hat{a}_{1/2}^+, \hat{a}_{1/2}$. In addition, much work along these lines makes the approximation of treating the bath as an assembly of classical HO's. In this case, one might hope that using a classical μ-wavelet oscillator bath can lead to improved results, since on the basis of the lower uncertainty product, the classical μ-wavelet bath should be a less severe approximation.

The appearance of the new raising and lowering operators, $\hat{a}_\mu^+, \hat{a}_\mu, \hat{a}_{1/2}^+, \hat{\hat{a}}_{1/2}, \hat{a}_{1/2}$ makes possible a greater variety of SUSY versions of harmonic oscillators coupled to a bath. Although the operator $\hat{\hat{a}}_{1/2}$ can be eliminated from the Hamiltonian expressions, it does not have to be. Additionally, it may prove interesting to introduce coupling terms that involve this new lowering operator, in addition to the other fractional lowering operator. In light of the fact that the operator pair $\hat{\hat{a}}_{1/2}, \hat{a}_{1/2}^+$ is associated with fermion dynamics, while the other pair of fractional ladder operators are associated with bosonic dynamics, this may give a simple means of exploring new types of coupling between fermionic and bosonic degrees of freedom. Another important role of the new lowering operators $\hat{a}_{1/2}$ and $\hat{A}$ is in the construction of new coherent and super-coherent states. Coherent states are important in many areas of physics. One of these is in the development of semi-classical approximations [15,46] to path integrals. In the Herman-Kluk [46] treatment, the coherent states are essentially used to filter the path integral kernel and it would be of interest to use the coherent states based on either μ-wavelets or sums of μ-wavelets to carry out such filtering. This would combine both the minimum uncertainty feature with the ability to smoothly and arbitrarily accurately approximate the ideal filter.

6.2 Possible experimental realization of μ-wavelet oscillator systems

Another interesting question is whether an experimental realization of these μ-wavelets and fractional ladder operators is possible for photons and atoms. For example, can one prepare optical supercoherent μ-wavelet states? In the past, there has been enormous interest in preparing so-called "Gauss-Hermite" light pulses (such solutions of Maxwell's equations were discovered by A. E. Siegman [47–49]). It is easily seen that the HDAFs are identical to Gauss-Hermite functions, although they were obtained in a radically different fashion and with radically different motivations [32–35]. The present approach is complementary to Siegman's work and the fact that significant progress has already been made in creating Gauss-Hermite pulses suggests that further study is justified [47–49]. For example, we stress that the "natural" creation and annihilation operators that manipulate such pulses had not been determined until the present work. In addition, it is intriguing whether one can prepare atoms in these new supercoherent states.

6.3 Similarity transformations

A fundamental axiom of quantum mechanics is that the allowed operators for the Hamiltonian (or any observable) must be Hermitian. It is curious that harmonic oscillation (see the Appendix on the classical limit of the μ-wavelet oscillator) also admits of a non-Hermitian Hamiltonian, whose spectrum remains exactly the same as the original Hamiltonian (due to the fact that the similarity transformation did not change the spectrum). It should be possible to carry out such similarity transformations for any standard Hermitian Hamiltonian system, leading to a non-Hermitian Hamiltonian operator. In the present case, the motivations for such a transformation are: (1) The states have favorable uncertainties in $\hat{x}$ and $\hat{k}$ compared to the original harmonic oscillator state. (2) It leads to new ladder operators that should describe a lower uncertainty harmonic oscillator bath. (3) The new μ-wavelet raising and lowering operators $\left\{\hat{\alpha}_\mu, \hat{\alpha}_\mu^+, \hat{a}_{1/2}, \hat{a}_{1/2}^+, \hat{A}, \hat{A}^+\right\}$ also lead to new coherent and super-coherent states, which again have favorable uncertainty products when expressed in terms of μ-wavelet oscillator states. (4) It demonstrates the strong connection between the μ-wavelet oscillator with SUSY-QM. In addition, the fact that there have already been attempts in the laser physics community to prepare such pulses also would argue for the value of such transformations [47–49].

6.4 The classical limit of the μ-wavelet harmonic oscillator

In the appendix to this paper, we have considered the classical limit of the μ-wavelet harmonic oscillator. We show explicitly that the classical dynamics remains harmonic, but there are significant changes in the phase behavior compared to the underlying standard harmonic oscillator. First, it is found that the frequency of the μ-wavelet harmonic oscillator is half that of the corresponding standard harmonic oscillator from which it was derived. Second, the phase of the momentum of the μ-wavelet oscillator increases monotonically, while that of the canonically conjugate position has both increasing and decreasing phase contributions. Thus, there is greater phase coherence to the dynamics of the momentum in the μ-wavelet oscillator. We speculate that this is a classical reflection of the fact that the μ-wavelet oscillator arises from squeezing the uncertainty in position subject to a constraint that the new minimum uncertainty state is not a Gaussian. In the constrained minimization procedure, one minimizes the product of the total uncertainty in position and the uncertainty in momentum only with respect to the added vector, $|\phi_j^\sigma\rangle$. Thus, in effect, the position is squeezed with the minimum increase in the uncertainty in the momentum.

Acknowledgments This paper is based on a thesis presented by Y. L. to the Physics Department of the University of Houston, in partial fulfillment of the requirements for the Ph. D. degree. We acknowledge helpful discussions with Dr. Bernhard G. Bodmann. Y.L. was supported under National Science Foundation Grant DMS-0406748 and R. A. Welch Foundation Grant E-0608. D.J.K. was supported under R. A. Welch Foundation Grant E-0608, National Science Foundation Grant DMS-0406748 and by the Texas Learning and Computation Center, University of Houston.

Appendix

In this appendix, we consider the classical limit of the μ-wavelet harmonic oscillator. It is important first to stress that in quantum systems, the similarity transformation, although not unitary, preserves the commutation relations among the various relevant raising and lowering operators (and between the position operator and its canonically conjugate momentum operator). On this basis, we conclude that the classical transformation corresponding to the quantum similarity transformation must preserve the canonical equations of motion (either in terms of the Hamiltonian or Lagrangian formalisms). This is crucial since it ensures that the classical limit of the quantum Hamiltonian will enable us to immediately obtain the correct equations of motion for the transformed position and momentum. As mentioned in the main text of this paper, the resulting classical transformation is an "extended canonical transformation". In fact, the classical limit of the μ-wavelet Hamiltonian is simply given as

$$\lim_{\hbar\to 0}\left(\frac{\hat{p}^2}{2m}-\frac{i\omega}{2}\hat{p}\hat{x}\right)=\lim_{\hbar\to 0}\hat{H}^{B}_{\mu-\text{wavelet}}=\lim_{\hbar\to 0}\hat{H}^{F(G)}_{\mu-\text{wavelet}}=H_{\mu-\text{wavelet}} \tag{80}$$

$$H_{\mu-\text{wavelet}}=\frac{\hat{p}^2}{2m}-\frac{i\omega}{2}\hat{p}\hat{x}. \tag{81}$$

Thus, one simply reverses the usual procedure used to obtain the quantum Hamiltonian from the classical one. Therefore, as $\hbar\to 0$, $\hat{x}$ and $\hat{p}$ become the ordinary position and momentum variables of Hamiltonian mechanics. Then the equations of motion are

$$\frac{\partial H_{\mu-\text{wavelet}}}{\partial p}=\dot{x} \tag{82}$$

$$\frac{\partial H_{\mu-\text{wavelet}}}{\partial x}=-\dot{p} \tag{83}$$

By Eq. 81, we find that

$$\dot{x} = \frac{p}{m} - \frac{i\omega}{2}x \tag{84}$$

$$-\dot{p} = -\frac{i\omega}{2}p \tag{85}$$

Equation (84) shows that the momentum canonically conjugate to x given by

$$p(t) = m\dot{x} + \frac{im\omega}{2}x(t) \tag{86}$$

It is trivial to solve Eq. 85 for $p(t)$ with result

$$p(t) = p(0)e^{i\omega t/2}. \tag{87}$$

Similarly we can solve for $x(t)$, to obtain

$$x(t) = x_1(0)e^{i\omega t/2} + x_2(0)e^{-i\omega t/2}, \tag{88}$$

where $p(0)$, $x_1(0)$, $x_2(0)$ are constants of integration. Of course, they are not all independent since only two constants of integration arise in solving the classical equations of motion. Evaluating Eq. 86 at $t = 0$ yields

$$p(0) = mi\omega x_1(0). \tag{89}$$

It is evident that the motion of the μ wavelet oscillator is harmonic. However, the motion is different from that of the original harmonic oscillator in that the new canonical momentum rotates in time with a monotonic increase in phase, while the time change of position has both increasing and decreasing phase contributions, just as for the original oscillator. In addition, the frequency of the motion is half that of the original oscillator. Explicitly, the original oscillator has the solutions

$$Q(t) = Q_1(0)e^{i\omega t} + Q_2(0)e^{i\omega t} \tag{90}$$

$$P(t) = im\omega[Q_1(0)e^{i\omega t} - Q_2(0)e^{-i\omega t}]. \tag{91}$$

It is then easy to display the explicit form of the extended canonical transformation relating the new variables in terms of the old. We observe that

$$e^{i\omega t} = \frac{im\omega Q(t) + P(t)}{2im\omega Q_2(0)} \tag{92}$$

$$e^{-i\omega t} = \frac{im\omega Q(t) - P(t)}{2im\omega Q_2(0)}. \tag{93}$$

It is then seen that

$$p(t) = im\omega x_1(0)\sqrt{\frac{im\omega Q(t) + P(t)}{2im\omega Q_2(0)}} \tag{94}$$

$$x(t) = x_1(0)\sqrt{\frac{im\omega Q(t) + P(t)}{2im\omega Q_2(0)}} + x_2(0)\sqrt{\frac{im\omega Q(t) - P(t)}{2im\omega Q_2(0)}}. \tag{95}$$

Finally, it is of interest to determine the classical μ-wavelet Lagrangian. We do this by the usual device of a Legendre transformation of the classical μ-wavelet Hamiltonian. Thus, we define

$$L_{\mu-\text{wavelet}} = p\dot{x} - H_{\mu-\text{wavelet}}. \tag{96}$$

Substituting Eq. 81 into 96, and using Eq. 86 to eliminate the variable $p(t)$, we obtain

$$L_{\mu-\text{wavelet}} = \frac{m\dot{x}^2}{2} + i\frac{m\omega}{2}\dot{x}x - \frac{m\omega^2}{8}x^2. \tag{97}$$

It is easily checked that this yields the correct canonical momentum:

$$\frac{\partial L_{\mu-\text{wavelet}}}{\partial \dot{x}} = m\dot{x} + \frac{im\omega}{2}x \equiv p. \tag{98}$$

Then the equation of motion is given by

$$\frac{\partial}{\partial t}\frac{\partial L_{\mu-\text{wavelet}}}{\partial \dot{x}} - \frac{\partial L_{\mu-\text{wavelet}}}{\partial x} = m\ddot{x} + \frac{im\omega}{4}x \equiv 0. \tag{99}$$

It is easy to verify that the solution is identical to that resulting from the classical Hamiltonian equations of motion.

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